M-math 2nd year Final Exam
Subject : Probability Theory

Time : 3.00 hours Max.Marks 50.

1. If $X = (X_1, \cdots, X_n)$ is a random vector, show that X is independent of the random variable Y if and only if $\lambda \cdot X$ (where the dot denotes inner product in $\mathbb{R}^n$) is independent of Y for all $\lambda \in \mathbb{R}^n$. (10)

2. Show that for each $T \in [0, \infty)$ and $t \in (0, \infty), T \neq t$,

$$\lim_{n \to \infty} e^{-nt} \sum_{k \leq nT} \frac{(nt)^k}{k!} = I_{(t,\infty)}(T).$$

Hint : Consider the sum of n independent Poisson random variables with a suitable parameter.

(10)

3. Let $\{X_n, n \geq 0\}$ be a simple symmetric random walk on $\mathbb{Z}$ with $X_0 \equiv 0$. For $a \in \mathbb{Z}$, let

$$\tau_a := \ inf\{k : X_k = a\}.$$

(a) Show that $\tau_a < \infty$ almost surely. Hint : Show that almost surely, $\limsup_{n \to \infty} X_n = \infty$.

(b) For $-a < 0 < b$ show that $P\{\tau_{-a} < \tau_b\} = \frac{b}{a+b}$.

(5 + 10)

4. Consider the Polya urn scheme, starting with k red and $n-k$ blue balls at time $i = 0$. At time $i \geq 1$ a ball is drawn at random and returned to the urn with an additional ball of the same colour. Let X_i = number of red balls at time i. Construct on some probability space a sequence of random variables $\{Y_i; i \geq 0\}$ having the same joint distribution as the sequence $(X_i)_{i \geq 0}$. (15)

5. Let $\{X_t, t \geq 0\}$ be a Poisson process with parameter $\lambda > 0$. Let $(\mathcal{F}_t)$ be its natural filtration. Let $0 \leq s < t < \infty$. Compute explicitly the regular conditional distribution of X_t given $\mathcal{F}_s$. (10)